19

The Cycloid

Why Study the Cycloid?

- **As a source of interesting problems**. It's the solution to the brachistochrone problem (curve of fastest descent), the tautochrone problem (isochronous motion), and can be used to square the circle.
- **Historical significance**. It played a pivotal role in the development of modern mathematics through the work of Galileo, Huygens, Bernoulli, Euler, and others. Most notably, the solution of the brachistochrone problem 'kicked off' the development of the calculus of variations.
- **Geometric richness**. The cycloid pushed mathematicians to refine tools for cusps, curvature, arc length, involutes, and differentiability.
- **Exact solvability in mechanics**. Motion along a cycloid can be treated analytically, providing rare closed-form results in classical dynamics.

19.1 The Cycloid

A cycloid is the curve generated by a point on the circumference of a circle rolling along a straight line [**Sim92, GH97**]. A cycloid whose generating circle has radius a is defined by the parametric equations

$$x = a(t - \sin t), \qquad y = a(1 - \cos t) \tag{19.1}$$

where t is the circle's angle of rotation, which starts at $t = 0$ at the origin. The equations can be easily obtained by considering the dimensions labeled in Fig. 19.1.

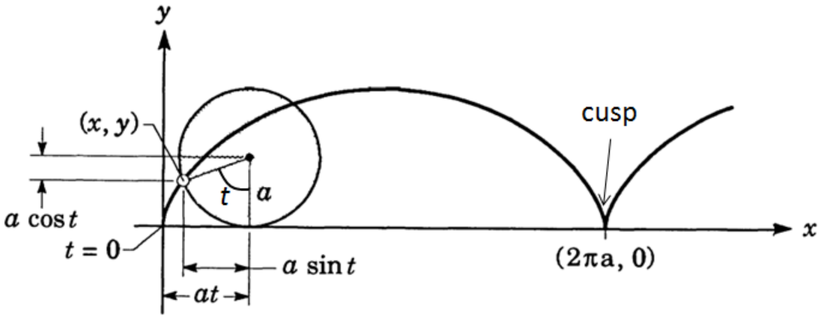


Figure 19.1. A Rolling Circle forms a Cycloid

From Equs. (19.1), we obtain the curve's derivative

$$y' = \frac{dy}{dx} = \frac{a \sin t \, dt}{a(1 - \cos t) \, dt} = \frac{\sin t}{1 - \cos t} = \frac{2 \sin \frac{1}{2} t \, \cos \frac{1}{2} t}{2 \sin^2 \frac{1}{2} t} = \cot \frac{1}{2} t.$$

Note that y' isn't defined for $t = 0, \pm 2\pi, \pm 4\pi$, etc, and those rotations correspond to *cusps* where the cycloid touches the x-axis, and its tangent becomes vertical.

Rather surprisingly, the area under one arch of a cycloid (e.g. between $x = 0$ and $2\pi a$) is three times the area of the rolling circle:

$$\begin{aligned} A &= \int_0^{2\pi a} y \, dx = \int_0^{2\pi} y \frac{dx}{dt} \, dt = \int_0^{2\pi} a(1 - \cos t) \, a(1 - \cos t) \, dt \\ &= a^2 \int_0^{2\pi} (1 - \cos t)^2 \, dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt \end{aligned}$$

So we need to evaluate:

$$\int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt$$

Split this into three parts:

$$\int_0^{2\pi} 1 \, dt - 2 \int_0^{2\pi} \cos t \, dt + \int_0^{2\pi} \cos^2 t \, dt$$

Consider each part in turn:

$$\int_0^{2\pi} 1 \, dt = 2\pi$$

and

$$\int_0^{2\pi} \cos t\, dt = [\sin t]_0^{2\pi} = \sin(2\pi) - \sin(0) = 0$$

For the third part, use the power reduction formula, $\cos^2 t = \frac{1+\cos(2t)}{2}$:

$$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2}\, dt$$

$$= \frac{1}{2} \cdot 2\pi + \frac{1}{2}\left[\frac{\sin(2t)}{2}\right]_0^{2\pi}$$

$$= \pi + 0 = \pi$$

Combining the results:

$$\int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt = 2\pi - 2(0) + \pi = 3\pi$$

The complete answer:

$$a^2 \int_0^{2\pi} (1 - \cos t)^2\, dt = \mathbf{3\pi a^2}$$

This result was first approximated by Galileo in 1599 through the comparison of the weights of circle and cycloid models. It was proved in 1634 by the French mathematician Roberval.

Another unusual result is that the length of one arch of the cycloid is four times the diameter of the rolling circle. Since $dx = a(1 - \cos t)dt$ and $dy = a \sin dt$, the arc length ds is given by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 = a^2\big[(1 - \cos t)^2 + \sin^2 t\big]\, dt^2 \\ &= 2a^2[1 - \cos t]\, dt^2 = 4a^2 \sin^2 \tfrac{1}{2}t\, dt^2, \\ ds &= 2a \sin \tfrac{1}{2}t\, dt. \end{aligned}$$

Therefore, the length of one arch (let's use $t = 0$ to 2π) is

$$L = \int_0^{2\pi} 2a \sin \tfrac{1}{2}t\, dt = \big[-4a \cos \tfrac{1}{2}t\big]_0^{2\pi} = \mathbf{8a}.$$

This result was first noted in 1658 by the English architect Christopher Wren.

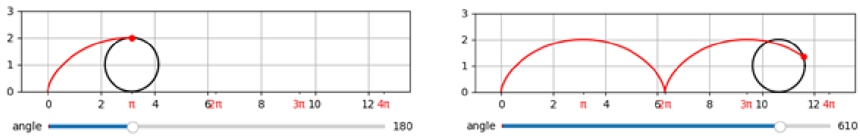


Figure 19.2. Rolling a Circle

19.1.1 Rolling a Circle. rollCycloid.py lets the user adjust the rotation angle of a circle, and the cycloid is drawn from the origin to a trace point located on the circle (see Fig. 19.2).

The circle's radius is $a = 1$ which explains why the cycloid cusps occur at 2π and 4π, and it peaks at $y = 2$.

The animation is updated by `update()` which translates the circle along the x-axis, and plots the cycloid:

```
def update(val):
  theta = math.radians(tSlider.val)
  Circle.set_xdata( xsCircleFn(theta))  # translate the circle
  ts = linspace(0, theta, max(2, int(50 * theta)))
  xs, ys = zip( *[cycloid(t, a) for t in ts])
  Cycloid.set_data(xs, ys)
  x, y = cycloid(theta, a)
  Point.set_data([x], [y])  # the tracer dot on the circle
  fig.canvas.draw_idle()

def cycloid(t, a):
  x = a * (t - math.sin(t))
  y = a * (1 - math.cos(t))
  return (x, y)
```

19.1.2 Examining Area and Arc Length. cycloidA.py lets the user change the rolling circle's radius (a), and draws the resulting cycloid between $t = 0$ and 27 radians. It calculates the area and arc length of a single arch of the curve in two ways – using the expressions $3\pi a^2$ and $8a$, and via numerical integration with Simpson's rule. Two screenshots are shown in Fig. 19.3.

The areas are produced by `calcArea()`:

```
def calcArea(a):
  analyticalArea = 3 * math.pi * a * a

  # For one arch, t goes from 0 to 2 pi
  ts = [i * (2*math.pi)/(nPts-1) for i in range(nPts)]
  xs, ys = cycloid(ts, a)
  # We need dx values for numerical integration
```

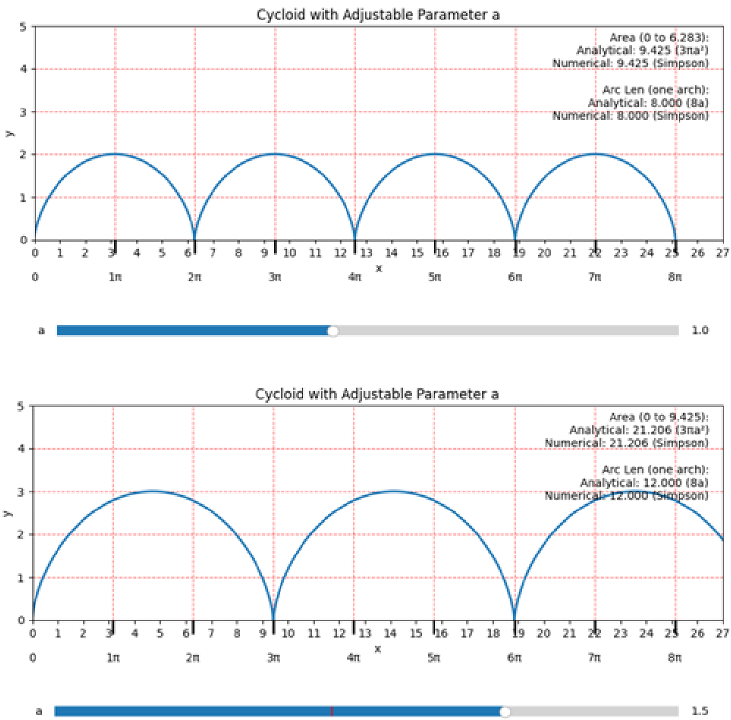


Figure 19.3. Cycloid Areas and arc lengths for Different Radii

```
    dxVals = [xs[i+1] - xs[i] for i in range(len(xs)-1)]
    # integrate y * (dx/dt) * dt, where dx/dt = a(1 - cos(t))
    dxdt = [a * (1 - math.cos(t)) for t in ts]
    integrand = [ys[i] * dxdt[i] for i in range(len(ys))]
    dt = (2 * math.pi) / (nPts - 1)
    numericalArea = simpson(integrand, dx=dt)
    return analyticalArea, numericalArea
```

Rather than code Simpson's rule ourselves (see section 6.11.4), we've imported `simpson()` from the SciPy module. It employs `integrand` to obtain the area under the curve as a collection of thin rectangular strips of dimension $y \times dx/dt$.

The arc lengths are calculated by `calcArcLen()`:

```
def calcArcLen(a):
    # arc length of one arch
    analyticalLen = 8 * a
```

```
  ts = [i * (2*math.pi)/(nPts-1) for i in range(nPts)]
  # Arc length numerical integral
  # Use cycloidDf() to get values for the derivatives
  integrand = []
  for t in ts:
    dx_dt, dy_dt = cycloidDf(t, a)
    integrand.append(math.sqrt(dx_dt**2 + dy_dt**2))
  dt = (2 * math.pi) / (nPts - 1)
  numericalLen = simpson(integrand, dx=dt)
  return analyticalLen, numericalLen


def cycloidDf(t, a):
  # Derivative of the cycloid
  dx_dt = a * (1 - math.cos(t))
  dy_dt = a * math.sin(t)
  return (dx_dt, dy_dt)
```

The numerical arc length is obtained by summing multiple small steps along the curve using $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.

19.1.3 Treating the Cycloid Like a Function.

Since the cycloid is defined parametrically:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t)$$

there's no explicit Cartesian form $y = f(x)$. Instead, we recover t from the x equation by root finding, and plug it into the y equation.

cycloidYT.py implements this approach, assuming that $a = 1$:

```
> python cycloidYT.py
Enter x value: 3.14159
y = 2.000000; t = 180.00 deg

> python cycloidYT.py
Enter x value: 1.570796
y = 1.673612; t = 132.35 deg

> python cycloidYT.py
Enter x value: 6.283185
y = 0.000075; t = 359.30 deg
```

Internally, the x equation is rearranged into zero form

$$a(t - \sin t) - x = 0$$

and SciPy's Brent's method is used to find the positive root for t. Brent is utilized instead of Newton–Raphson because it doesn't require derivatives and handles cycloid cusps where $dx/dt = 0$.

The relevant code is in `cycloidYTfromX()`:

```
def cycloidYTfromX(x, a):
  archIndex = math.floor(x / archWidth)
  if archIndex != 0:
    xArch = x - archIndex * archWidth
      # handles negative and large x's
  else:
    xArch = x
  if abs(xArch) < 1e-12 or abs(xArch - archWidth) < 1e-12:
    return 0, 0      # cusp handling
  else:
    tMin = 0
    tMax = 2*math.pi
    t = brentq(cycloidT, tMin, tMax, args=(xArch, a))
    y = a * (1 - math.cos(t))
    return y, t + archIndex*2*math.pi


def cycloidT(t, x, a):
  return a * (t - math.sin(t)) - x
```

19.2 The Tautochrone

A tautochrone is the curve for which the time taken by a bead sliding without friction in uniform gravity along the curve to its lowest point is independent of its starting point.

The curve was first studied by Huygens after he realized that a pendulum, which obviously swings in a circular arc, keeps different times depending on how far the pendulum is initially pulled away from the vertical. He investigated what would happen if the arc was changed to an inverted cycloid, and discovered that the pendulum bob would swing down from any starting point to the bottom in the same amount of time. In other words, an inverted cycloid is a tautochrone.

One way to formalize this is to turn Fig. 19.1 upside down, as in Fig. 19.4. This points the y-axis in the direction of the gravitational force but makes the downward y-coordinates positive, conveniently leaving the cycloid equations unchanged.

The kinetic energy of the bead is initially zero, since it's at rest. The work done by gravity in moving the bead from $(0,0)$ to some point (x,y) is mgy, and this must equal the change in kinetic energy. That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the velocity of the bead when it reaches (x,y) is

$$v = \sqrt{2gy}.$$

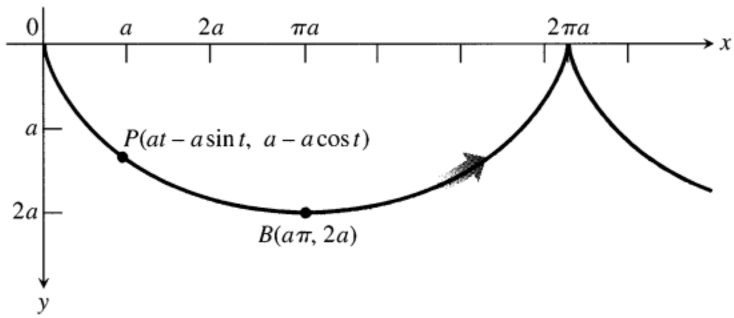


Figure 19.4. An Inverted Cycloid

That is,

$$\frac{ds}{dt} = \sqrt{2gy},$$

where ds is the arc length differential along the bead's path, or

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\, dx}{\sqrt{2gy}}.$$

The time T_f it takes the bead to slide along a curve $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\, dx.$$

What curve $y = f(x)$, if any, minimizes the value of this integral? For the cycloid, T_f takes the form

$$T_{\text{cycloid}} = \int_{x=0}^{x=a\pi} \sqrt{\frac{dx^2 + dy^2}{2gy}}.$$

From Equs. (19.1):

$$dx = a(1 - \cos t)\, dt, \qquad dy = a \sin t\, dt,$$

and so

$$T_{\text{cycloid}} = \int_{t=0}^{t=\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(1 - \cos t)}}\, dt$$
$$= \int_0^{\pi} \sqrt{\frac{a}{g}}\, dt = \pi\sqrt{\frac{a}{g}}.$$

Thus, the amount of time it takes the friction-less bead to slide down the cycloid from O to the bottom at B is $\pi\sqrt{a/g}$.

Suppose that instead of O we start the bead at some lower point, (x_0, y_0). The bead's velocity at (x, y) is now

$$v = \sqrt{2g(y - y_0)} = \sqrt{2ga(\cos t_0 - \cos t)}$$

since $y = a(1 - \cos t)$. Accordingly, the time required for the bead to slide from (x_0, y_0) down to B is

$$\begin{aligned} T &= \int_{t_0}^{\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(\cos t_0 - \cos t)}}\, dt = \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{1 - \cos t}{\cos t_0 - \cos t}}\, dt \\ &= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{2\sin^2(t/2)}{(2\cos^2(t_0/2) - 1) - (2\cos^2(t/2) - 1)}}\, dt \\ &= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \frac{\sin(t/2)\, dt}{\sqrt{\cos^2(t_0/2) - \cos^2(t/2)}} \end{aligned}$$

Apply the substitution $u = \cos(t/2)$ (and $-2\, du = \sin(t/2)\, dt$), and set $c = \cos(t_0/2)$:

$$T = \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \frac{-2\, du}{\sqrt{c^2 - u^2}}$$

Integrating:

$$\begin{aligned} T &= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1} \frac{u}{c} \right]_{t_0}^{\pi} \\ &= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1} \frac{\cos(t/2)}{\cos(t_0/2)} \right]_{t_0}^{\pi} \\ &= 2\sqrt{\frac{a}{g}} (-\sin^{-1} 0 + \sin^{-1} 1) = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

This is the same amount of time that it took the bead to slide from O to B, so beads starting simultaneously from different points on the inverted cycloid in Fig. 19.4 will all reach the bottom of the curve at B concurrently.

19.2.1 Simple Harmonic Motion (SHM). T is the descent time from O to the bottom of the cycloid at B. This implies that the oscillation time (or period), from O to the other extreme, and back to O, is $4 * T = 4\pi\sqrt{a/g}$, which is often written as

$$T_{\text{osc}} = 2\pi\sqrt{\frac{4a}{g}}.$$

In general, any motion that is simple harmonic relates its period T_p to its angular frequency ω by

$$T_p = \frac{2\pi}{\omega}.$$

So the cycloid exhibits SHM, and we can equate

$$\omega = \sqrt{\frac{g}{4a}}.$$

19.2.2 Animating the Cycloid as a Tautochrone. SHM is also characterized by the differential equation:

$$\frac{d^2s}{dt^2} + \omega^2 s = 0,$$

where s is the object's displacement from its equilibrium position. With initial conditions $s(0) = s_0$ and $s'(0) = 0$, this equation has the solution:

$$s(t) = s_0 \cos \omega t$$

This gives us a simple way to implement the movement of multiple beads down an inverted cycloid in tautochrone.py, as shown in Fig. 19.5.

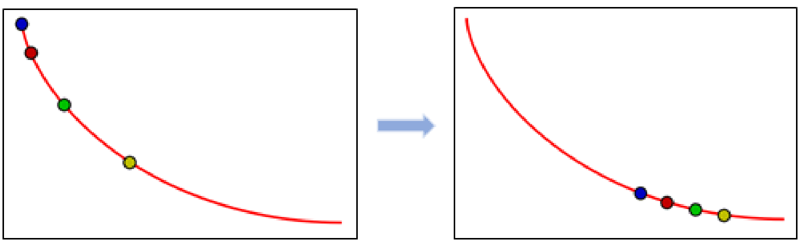


Figure 19.5. A Cycloid as a Tautochrone

Each bead is initialized with a starting t value and color.

```
beads = [
  {'theta':1.0, 'color':'#0000c0'},
  {'theta':0.8, 'color':'#c00000'},
  {'theta':0.6, 'color':'#00c000'},
  {'theta':0.4, 'color':'#c0c000'}
]
```

Inside `animate()`, which is called for each animation frame, each bead's 'theta' is combined with the current `w` to calculate its (x, y) position.

```
def animate(fno):
  w = fno/(nFrames-1)  # 0 to 1
  for i, bead in enumerate(beads):
    theta = -bead['theta']*math.pi * math.cos(w * math.pi/2)
    xc, yc = cycloid(theta, a)
    x = h/2*xc + x0
```

```
    y = h/2*yc + y0
    # (x0,y0) is the top point; h is the window height
    beadPts[i].set_data([x], [y])
  return beadPts
```

19.3 A Cycloid's Involute

Once Huygens had established the cycloid's tautochrone property, how could he arrange for a clock's pendulum to move along a cycloidal, rather than a circular, path? One solution is to suspend the pendulum from the cusp between two inverted cycloids with its rope's length equal to the cycloid's semi-arch (see Fig. 19.6). This constrains the pendulum's path to follow a cycloid.

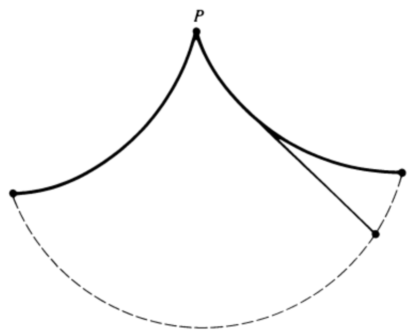


Figure 19.6. An Cycloidal Pendulum

Huygens termed this curve an *involute*: the locus of the end of a taut string as it unwraps itself from a cycloid surface (or wraps itself over such as surface). The equation for the involute is obtained as follows. Start with the parametric form of the surface curve

$$\mathbf{C}(t) = \big(x(t),\, y(t)\big),$$

where $\mathbf{C}'(t) \neq \mathbf{0}$, and define its arc length between a starting angle t_0 and t:

$$s(t) = \int_{t_0}^{t} \|\mathbf{C}'(u)\|\, du,$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{C}'(t)}{\|\mathbf{C}'(t)\|},$$

which allows the involute to be expressed as

$$\mathbf{I}(t) = \mathbf{C}(t) - s(t)\,\mathbf{T}(t).$$

This is depicted in Fig. 19.7. The end of the (purple) string initially touches the surface curve **C**() at $x = \pi a$, the peak of the cycloid's arch, and is attached at $x = 2\pi a$, in the cusp between two arches. Currently, the string has unwrapped so it's tangent is at angle t, and the tangent segment's length is $s(t)$. The involute traced out by the end of the string is **I**() (the red curve).

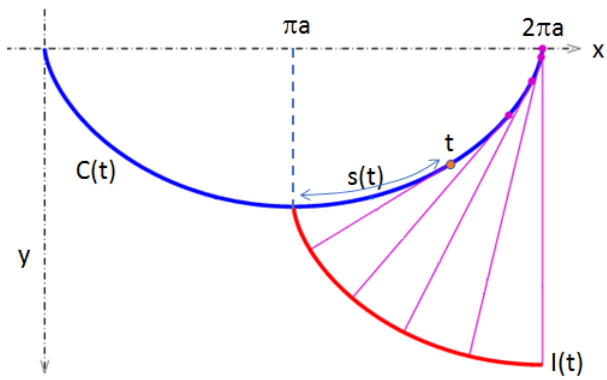


Figure 19.7. An Involute of a Cycloid

We can plug cycloid-specific details into $\mathbf{I}(t)$:

$$\mathbf{C}(t) = \big(a(t - \sin t),\ a(1 - \cos t)\big), \qquad \mathbf{C}'(t) = a\big(1 - \cos t,\ \sin t\big),$$

and

$$\|\mathbf{C}'(t)\| = 2a \sin \frac{t}{2}.$$

The arc length of the unwrapped string, $s(t)$, from $t = \pi$ to the current angle, is

$$s(t) = \int_{\pi}^{t} 2a \sin \frac{u}{2}\, du = 2a\Big[-2\cos \frac{u}{2}\Big]_{u=\pi}^{u=t}$$
$$= -4a \cos \frac{t}{2} - \Big(-4a \cos \frac{\pi}{2}\Big) = -4a \cos \frac{t}{2}.$$

The unit tangent is

$$\mathbf{T}(t) = \frac{\mathbf{C}'(t)}{\|\mathbf{C}'(t)\|} = \frac{a(1 - \cos t, \sin t)}{2a \sin \frac{1}{2}} = \Big(\sin \frac{t}{2},\ \cos \frac{t}{2}\Big),$$

by utilizing half angles identities to rewrite the numerator.

We formulate $\mathbf{I}(t) = \mathbf{C}(t) - s(t)\mathbf{T}(t)$, by considering its x and y parts separately:

$$I_x(t) = a\big(t - \sin t\big) - \big(-4a\cos\frac{t}{2}\big)\sin\frac{t}{2} = a\big(t - \sin t\big) + 4a\cos\frac{t}{2}\sin\frac{t}{2}$$
$$= a\big(t - \sin t\big) + 2a\sin t = a\big(t + \sin t\big),$$

$$I_y(t) = a\big(1 - \cos t\big) - \big(-4a\cos\frac{t}{2}\big)\cos\frac{t}{2} = a\big(1 - \cos t\big) + 4a\cos^2\frac{t}{2}$$
$$= a\big(1 - \cos t\big) + 2a\big(1 + \cos t\big) = a\big(3 + \cos t\big).$$

So:

$$\mathbf{I}(t) = \big(a(t + \sin t),\ a(3 + \cos t)\big)$$

How does this compare to the surface cycloid equation, $\mathbf{C}(t)$, with t replaced by ϕ:

$$\mathbf{C}(\phi) = \big(a(\phi - \sin\phi),\ a(1 - \cos\phi)\big).$$

Set $\phi = t + \pi$, and simplify $\mathbf{C}()$:

$$\mathbf{C}(t + \pi) = a\big((t + \pi) - \sin(t + \pi),\ 1 - \cos(t + \pi)\big)$$
$$= a\big(t + \pi + \sin t,\ 1 + \cos t\big).$$

Equate the components of $\mathbf{I}()$ and $\mathbf{C}()$:

$$I_x(t) = a(t + \sin t) = a(t + \pi + \sin t) - a\pi = C_x(t + \pi) - a\pi,$$

$$I_y(t) = a(3 + \cos t) = a(1 + \cos t) + 2a = C_y(t + \pi) + 2a$$

or,

$$\mathbf{I}(t) = \mathbf{C}(t + \pi) + (-a\pi,\ 2a)$$

This shows that the involute $\mathbf{I}(t)$ is a cycloid, with its t angle is rotated by π and translated by $(-a\pi,\ 2a)$) away from $\mathbf{C}(t)$.

19.3.1 Drawing a Cycloid Involute. involuteCycs.py unwraps two strings from the crests of adjacent cycloids towards a center line above the intermediate cusp (see Fig. 19.8). As the strings move, line segments are drawn at their current tangents with lengths equal to the arc distance from the crests to those tangents. The coordinates of the ends of the moving segments are collected and plotted, forming two halves of a new cycloid.

During the animation, `update()` draws each frame by calling `tangentSeg()` twice to get the coordinates for the ends of the segments. To explain `tangentSeg()`, we'll focus on how (xEnd, yEnd) is calculated for the left-hand string, as shown in Fig. 19.9.

It's a little hard to see but `tangentSeg()` implements a version of

$$\mathbf{I}(t) = \mathbf{C}(t) - s(t)\,\mathbf{T}(t)\ :$$

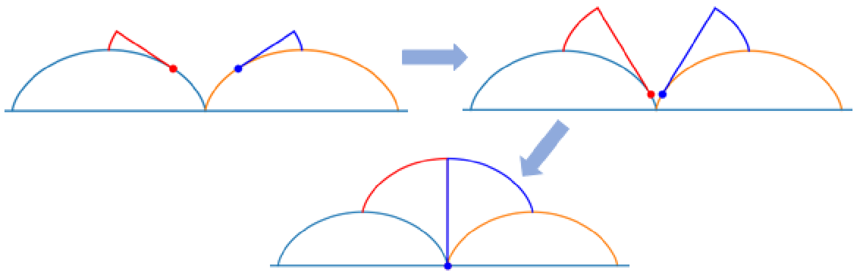


Figure 19.8. Drawing a Cycloid Involute

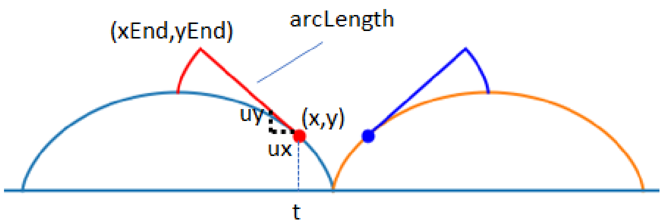


Figure 19.9. Calculating a Segment End Coordinate

```
def tangentSeg(t, a, xOffset, dir):
  x, y = cycloid(t, a, xOffset)     # C(t)
  dxdt, dydt = cycloidDf(t, a)      # C'(t)
  speed = math.hypot(dxdt, dydt)    # |C'(t)|
  if speed == 0.0:
    return [x, x], [y, y], x, y

  ux = dxdt / speed      # T(t) for x and y
  uy = dydt / speed
  if dir < 0:
    ux = -ux
    uy = -uy
  arcLength = 4.0 * a * abs(math.cos(t / 2.0))  # s(t)
  xEnd = x + arcLength * ux     # I(t) for x and y
  yEnd = y + arcLength * uy
  return [x, xEnd], [y, yEnd], xEnd, yEnd
```

$\mathbf{C}(t)$ is implemented by `cycloid()`, while $\mathbf{T}(t) = \mathbf{C}'(t)/\|\mathbf{C}'(t)\|$ employs `cycloidDf()` and math.hypot(). $s(t)$, the arc length, must be $4a$ or less since the unwinding string only extends half-way around the cycloid (between $x = \pi a$ and $2\pi a$ on the left hand side).

`xOffset` and `dir` are used to generalize the function so it can be employed for the string wrapped over the right-hand cycloid. The relevant lines in `update()` are:

```
lx, ly, lxEnd, lyEnd = tangentSeg(tLeft, a, 0.0, dir=-1)
rx, ry, rxEnd, ryEnd = tangentSeg(tRight, a, 2.0*math.pi, dir=1)
```

19.4 The Brachistochrone

In 1696, Johann Bernoulli conceived and solved the brachistochrone problem, and published the problem (but not the solution) as a challenge to other mathematicians. The problem is this: among all smooth curves in a vertical plane that join a point P_0 to a lower point P_1, not directly below it, find the curve along which a particle will slide from P_0 to P_1 in the shortest possible time.

Following Huygens' approach, we can think of the particle as a bead of mass m sliding down a friction-less wire, with mg the only force acting upon it. Locate P_0 at the origin and set $P_1 = (x_1, y_1)$ as in Fig. 19.10. Note that P_1 does not need to be the bottom of the cycloid (i.e. point B in Fig. 19.4).

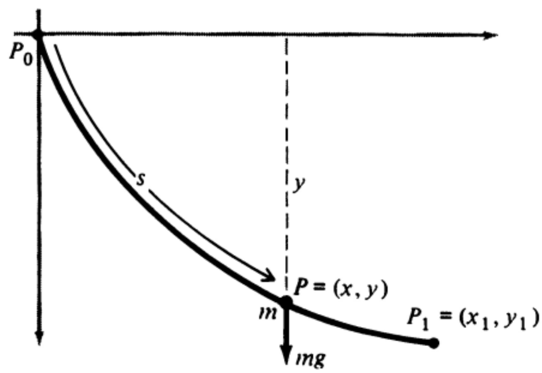


Figure 19.10. A Brachistochrone Bead at P

The bead is released from rest at P_0, so its initial velocity and kinetic energy are zero. The work done by gravity in pulling it down to an arbitrary point $P = (x, y)$ is mgy, which must equal the increase in its kinetic energy. So $\frac{1}{2}mv^2 = mgy$, and therefore

$$v = \frac{ds}{dt} = \sqrt{2gy}.$$

This can be written as

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\,dx}{\sqrt{2gy}}.$$

The total time T_1 required for the bead to slide down the wire from P_0 to P_1, will depend on the wire's shape as specified by $y = f(x)$, so that

$$T_1 = \int dt = \int_0^{x_1} \sqrt{\frac{1 + (y')^2}{2gy}}\,dx.$$

The brachistochrone problem therefore amounts to finding a curve $y = f(x)$ that passes through P_0 and P_1 that minimizes this integral.

We start by considering an apparently unrelated problem in optics. Fig. 19.11 illustrates a ray of light traveling from A to P with constant velocity v_1. Upon entering a denser (gray) medium, it travels from P to B with a smaller velocity v_2.

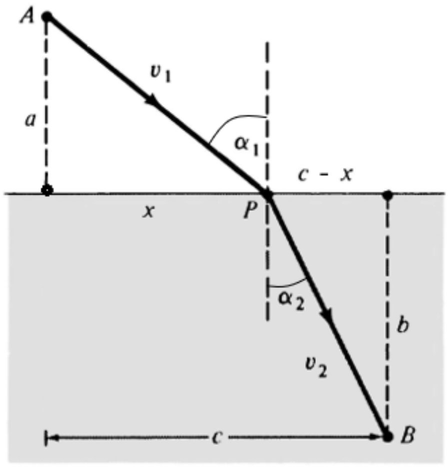


Figure 19.11. The Refraction of Light

The total time T required for the journey is

$$T = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c - x)^2}}{v_2}.$$

If we assume that the ray of light is able to select its path from A to B in such a way as to minimize T, then $dT/dx = 0$. Then the two parts of that derivative can be equated:

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{c - x}{v_2\sqrt{b^2 + (c - x)^2}},$$

By referring to Fig. 19.11, the distances ratios can be replaced by sines involving α_1 and α_2:

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}.$$

This is *Snell's law of refraction*, and the assumption that light travels from one point to another along the path requiring the shortest time is *Fermat's principle of least time.*

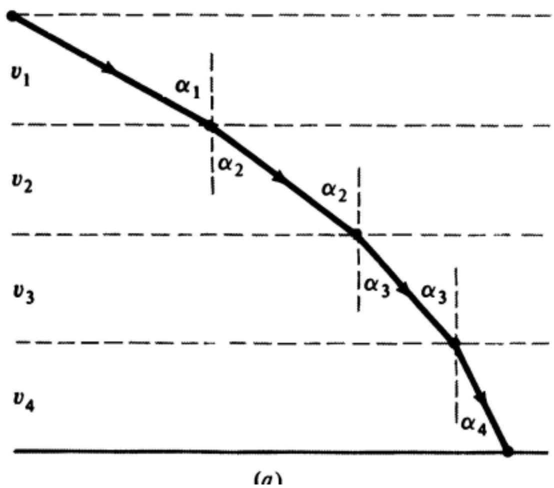


Figure 19.12. Refraction Angles

The velocity of light is constant within a layer in Fig. 19.12, but decreases as it passes down through each subsequent denser layer, and is refracted more and more towards the vertical. When Snell's law is applied at the boundaries between the layers, we obtain

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = \frac{\sin\alpha_3}{v_3} = \frac{\sin\alpha_4}{v_4}.$$

As we make these layers thinner and more numerous, in the limit the velocity of light decreases continuously as the ray descends, and we get

$$\frac{\sin\alpha}{v} = c$$

Returning to the brachistochrone problem, let's introduce the coordinate scheme shown in Fig. 19.13 and assume that the bead (like the ray of light) is capable of selecting the path from P_0 to P_1 that requires the shortest possible travel

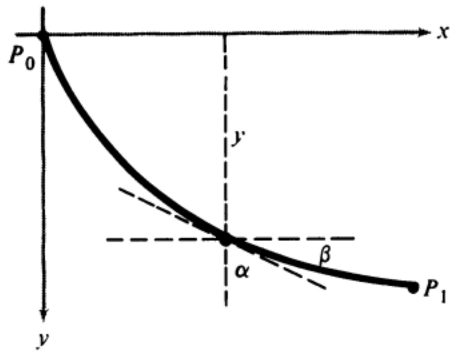


Figure 19.13. Refraction for the Brachistochrone

time. The argument given above yields the same result for the sliding bead

$$\frac{\sin\alpha}{v} = c$$

We also have

$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}}$$
$$= \frac{1}{\sqrt{1+(y')^2}}.$$

Combining these equations, and the earlier velocity equality $v = \sqrt{2gy}$, produces

$$y[1+(y')^2] = k.$$

Replace y' by dy/dx, and separate the variables

$$dx = \sqrt{\frac{y}{k-y}}\,dy,$$

so

$$x = \int \sqrt{\frac{y}{k-y}}\,dy.$$

Employ the substitution $u^2 = y/(k-y)$ so that

$$y = \frac{ku^2}{1+u^2} \quad \text{and} \quad dy = \frac{2ku}{(1+u^2)^2}\,du,$$

resulting in

$$x = \int \frac{2ku^2}{(1+u^2)^2}\,du.$$

The trigonometric substitution $u = \tan\phi$, and its derivative $du = \sec^2\phi\, d\phi$, let us write this as

$$\begin{aligned} x &= \int \frac{2k\,\tan^2\phi\,\sec^2\phi}{(1+\tan^2\phi)^2}\,d\phi \\ &= 2k\int \frac{\tan^2\phi}{\sec^2\phi}\,d\phi = 2k\int \sin^2\phi\,d\phi \\ &= k\int (1-\cos 2\phi)\,d\phi \\ &= \frac{1}{2}k(2\phi - \sin 2\phi). \end{aligned}$$

The constant of integration is zero because $y = 0$ when $\phi = 0$, and since P_0 is at the origin, we also have $x = 0$ when $\phi = 0$.

The formula for y is

$$y = \frac{k\tan^2\phi}{\sec^2\phi} = k\sin^2\phi = \frac{1}{2}k(1-\cos 2\phi).$$

Simplifying these equations by writing $a = \frac{1}{2}k$ and $\theta = 2\phi$, leads to

$$x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta).$$

In other words, the curve that produces the shortest travel time for the Brachistochrone problem is the cycloid

19.4.1 Animating the Brachistochrone.

brachAnim.py visualizes and compares three different paths (a straight line, an arc of a circle, and a cycloid) for a bead sliding under gravity from point (0,0) to an endpoint.

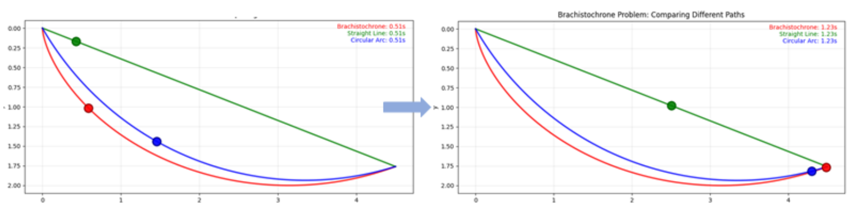


Figure 19.14. The Brachistochrone Race

The animation demonstrates that the cycloid, despite being longer than a circular arc between the same two points, causes the bead to reach the endpoint faster than both the straight line and arc. However, the cycloid bead only *just* beats the bead sliding along the arc, so it's not surprising that early investigators, such as Galileo, concluded that the arc was fastest.

It's possible to pause/resume the animation by pressing the space bar, which allows the user to compare the positions of the beads during their journey. One interesting aspect is that the circular bead (the blue dot) is ahead of the cycloid bead (the red dot) for much of the trip (see Fig. 19.14).

It's important to remember that `brachAnim.py` is not a real-world experiment (e.g. unlike the one at `https://mathsmodels.co.uk/2021/06/01/Brachistochrone2/`). For example, no account is made of friction or air resistance, and the course of the bead's path along the arc is only approximate because the exact time equation is an elliptic integral of the form

$$t(\theta) = \sqrt{\frac{R}{2g}} \int_{\theta_0}^{\theta} \frac{d\phi}{\sqrt{\cos\phi - \cos\theta_0}},$$

where R is the circle's radius, and the bead travels from angular position θ_0 to θ. There's no closed form solution for this expression, so numerical methods must be used. My code employs time = distance / velocity, where the total distance is the sum of small segments along the arc, and the current velocity is $\sqrt{2gh}$ where h is the vertical drop from the starting point. The details can be found in `calculateCircularArc()` in brachAnim.py.

Another issue is what curvature should be used for the arc since it's only constrained by two points. My code employs a fixed relationship to transform the chord length between the start and end points into the circle's radius.

The bead's journey time along the cycloid is much simpler; it's based on

$$T = \pi\sqrt{\frac{a}{g}},$$

resulting in:

```
_, thetaEnd = cycloidYTfromX(xEnd, a)
t = thetaEnd * math.sqrt(a / G)
brCurveX, brCurveY = cycloidPts(a, thetaEnd, 500)
```

The difficult part is determining the angle `thetaEnd` of the endpoint (which is not necessarily π) but I reuse `cycloidYTfromX()` from section 19.1.3 for that task.

19.5 Epicycloids

An epicycloid is a curve generated by a point on the circumference of a circle which is rolling on the *outside* of a fixed circle (see Fig. 19.15). In other words, it's the cycloid again but with the flat surface replaced by a circle [**Mao20**].

More complicated curves can be generated by having a third circle roll on the second, a fourth on the third, etc. Epicycloids were introduced by the ancient Greeks and, until the 16th century, formed the basis for ideas about the paths of the Moon and planets.

An epicycloid whose fixed and rolling circles have radii R and r respectively is described by the parametric equations:

$$x = (R + r)\cos t - r\cos(\frac{R+r}{r}t) \qquad y = (R + r)\sin t - r\sin(\frac{R+r}{r}t).$$

t is the angle formed by the x-axis and the ray from the center of the fixed circle to the point of contact with the rolling circle, as depicted in Fig. 19.15.

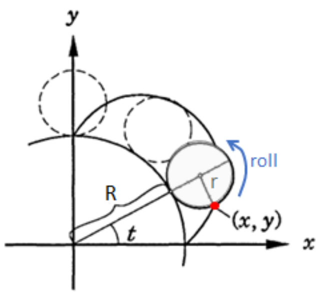


Figure 19.15. The Epicycloid Circles

If the radii of the fixed circle and rolling circle are equal ($R = r$) then a complete tour by the rolling circle around the fixed one will form a single arc. If $R = 2r$ then the tour will include two arcs, and when $R = nr$ the tour will contain n arcs. Between the arcs, the curve will form cusps when the rolling circle's tracer point touches the fixed circle.

epicycloid.py asks the user to enter a float representing R/r, which is used to calculate the radius of the small circle (r) since the large circle is a fixed size ($R = 4$).

The plotting uses a (verbose) translation of the maths:

```
def epicycloidPt(bigR, smallR, t):
  x = (bigR + smallR) * math.cos(t) \
             - smallR * math.cos((bigR + smallR)/smallR * t)
  y = (bigR + smallR) * math.sin(t) \
             - smallR * math.sin((bigR + smallR)/smallR * t)
  return x, y
```

Fig. 19.16 shows four runs of `epicycloid.py`, including a cardioid (one cusp; $R/r = 1$) and a nephroid (two cusps; $R/r = 2$).

19.5.1 From Epicycloid to Epicycle. The parametric equations can be better visualized by increasing the size of the fixed circle, as in Fig. 19.17, so the rotating tracer point (the red dot) on the circumference of the rolling circle can be specified relative to the circumference of the fixed circle. In this version, the smaller circle is usually called the *epicycle* and the larger circle the *deferent*.

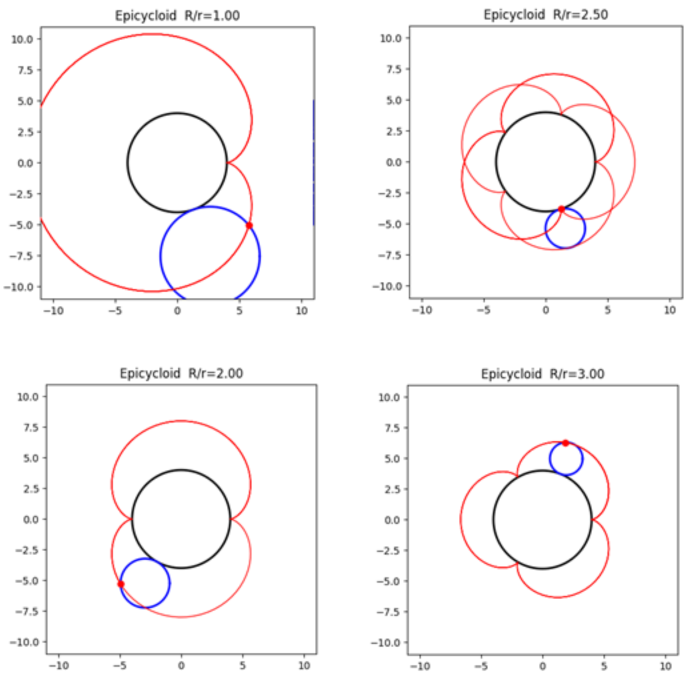


Figure 19.16. Four Epicycloids

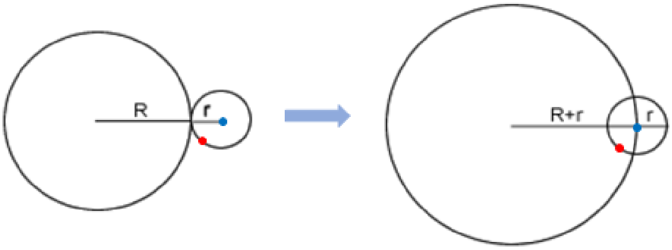


Figure 19.17. From Epicycloid to Epicycle

It's also useful to redefine the epicycle's equations in terms of angular speed. Let

$$\mathbf{r}(t) = (R + r)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + r\begin{pmatrix}-\cos\left(\frac{R+r}{r}t\right)\\ -\sin\left(\frac{R+r}{r}t\right)\end{pmatrix}$$

or:

$$\mathbf{r}(t) = R_{\text{def}} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} + r_{\text{epi}} \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \end{pmatrix}$$

where:

- R_{def} is the radius of the deferent,
- r_{epi} is the radius of the epicycle,
- ω is the angular speed of the deferent,
- Ω is the angular speed of the epicycle relative to the deferent,

The angular velocities are related by:

$$\frac{\Omega}{\omega} = -\frac{R + r}{r}$$

By adding more epicycles, with varying speeds and sizes, we can formulate increasingly complex curves:

$$x(t) = \sum_i R_i \cos(\omega_i t + \phi_i)$$
$$y(t) = \sum_i R_i \sin(\omega_i t + \phi_i)$$

ϕ_i is how much circle i is initially rotated, and is called a **phase offset**. If we don't utilize phase offsets, then we can only describe epicycles where all the circles start neatly aligned.

19.5.2 From Epicycles to Discrete Fourier Transforms. Instead of thinking about points along the x and y axes, imagine them as complex numbers using the real and imaginary axes:

$$p_j = x_j + iy_j$$
$$z(t) = x(t) + iy(t) = \sum_j^N R_j(\cos(\omega_j t + \phi_j) + i \sin(\omega_j t + \phi_j))$$

Now make use of Euler's formula:

$$z(t) = \sum_j^N R_j e^{i\,\omega_j t + \phi_j}$$

Even better, permit X_j be a complex number:

$$z(t) = \sum_j^N X_j e^{i\omega_j t}$$

This form almost matches the *Discrete Fourier Transform* (DFT) [**Kre10**]. This shouldn't be too surprising since DFTs essentially allows us to split any periodic function into a series of connected sinusoidal functions.

19.6 Hypocycloids

A hypocycloid is a curve generated by a point on the circumference of a circle rolling on the *inside* of a fixed circle [**Mao20**]. A hypocycloid whose fixed and rolling circles have the radii R and r respectively has the parametric equations

$$x = (R-r)\cos t - r\cos(\frac{R-r}{r}t) \qquad y = (R-r)\sin t - r\sin(\frac{R-r}{r}t).$$

Once again, t is the angle formed by the x-axis and the ray from the center of the fixed circle to the point of contact with the rolling circle, as in Fig. 19.18.

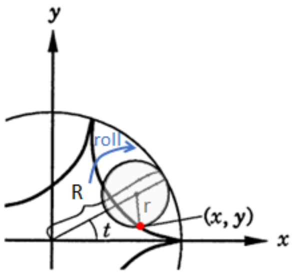


Figure 19.18. The Hypocycloid Circles

In a similar manner to the epicycloid, the hypocycloid has a cusp where its tracer point meets the fixed circle, and describes two arcs when the radius of the fixed circle is twice the length of the radius of the rolling circle ($R = 2r$), and n arcs when $R = nr$.

hypocycloid.py asks the user to enter a float representing R/r, which is used to calculate the radius of the small circle (r) since the large circle is a fixed size ($R = 5$).

The code based on the maths:

```
def hypocycloidPt(bigR, smallR, t):
  x = (bigR - smallR) * math.cos(t) \
      + smallR * math.cos((bigR - smallR) / smallR * t)
  y = (bigR - smallR) * math.sin(t) \
      - smallR * math.sin((bigR - smallR) / smallR * t)
  return x, y
```

Fig. 19.19 shows six runs of `hypocycloid.py`, and illustrates a few interesting cases: a straight line when $R/r = 2$, and a deltoid and astroid. Note that an astroid appears when $R/r = 4$ (as expected) *and* also when $R/r = 4/3$.

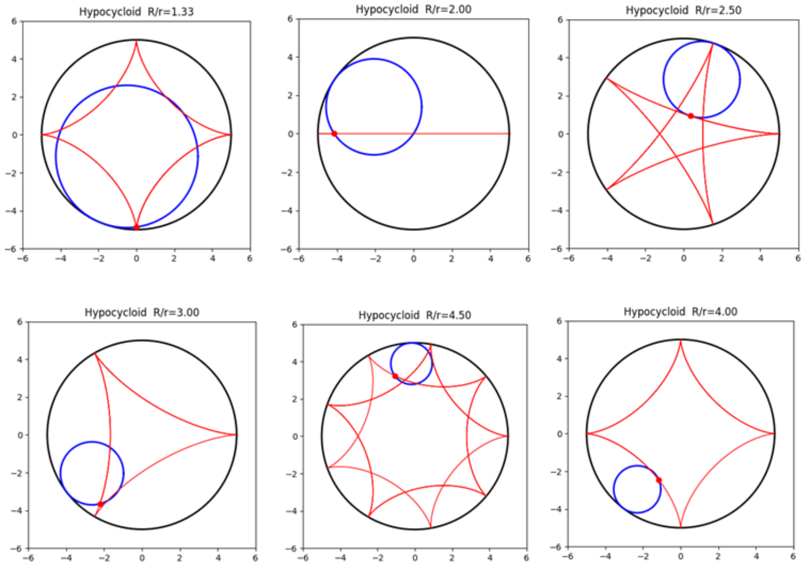


Figure 19.19. Six Hypocycloids

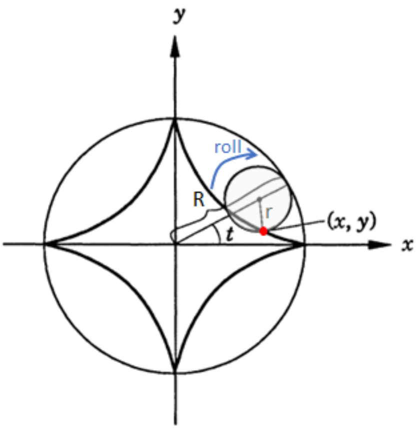


Figure 20.20. The Astroid

19.6.1 The Astroid. The astroid is a hypocycloid whose rolling circle has a diameter one-fourth that of the fixed circle (see Fig. 19.20).

The formula for the astroid sets $R = 4r$ in the equations for the hypocycloid:

$$x = (4r - r)\cos t + r\cos\left(\frac{4r-r}{r}t\right) = r(3\cos t + \cos 3t),$$
$$y = (4r - r)\sin t - r\sin\left(\frac{4r-r}{r}t\right) = r(3\sin t - \sin 3t).$$

With the help of

$$\cos 3t = 4\cos^3 t - 3\cos t; \quad \sin 3t = 3\sin t - 4\sin^3 t,$$

we find

$$x = r(3\cos t + \cos 3t) = 4r\cos^3 t$$
$$y = r(3\sin t - \sin 3t) = 4r\sin^3 t,$$

or

$$x = R\cos^3 t, \qquad y = R\sin^3 t.$$

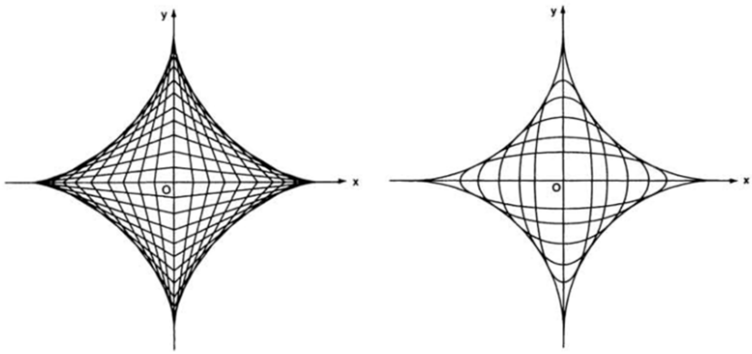
These equations are implemented in astroid.py as:

```
def astroid(a, nPts):
  ts = [2 * math.pi * i / nPts for i in range(nPts + 1)]
  xs = [a * (math.cos(t) ** 3) for t in ts]
  ys = [a * (math.sin(t) ** 3) for t in ts]
  return xs, ys
```

`astroid.py` generates the same shape as the one shown in the bottom right of Fig. 19.19. But what about the top-left example, where $R/r = 4/3$? This only changes the very last step of the equation's derivation given above since now $3R = 4r$. This affects the drawing behavior, but its only noticeable in the animation – the rolling circle for $R/r = 4/3$ takes 3 tours around the fixed circle to complete the astroid, whereas when $R/r = 4$ the much smaller rolling circle needs just a single tour. In general, a hypocycloid with ratio R/r (in reduced form) completes its pattern after the rolling circle has made r tours around the fixed circle.

The astroid has some remarkable properties. For example, all of its tangent lines are the same length R between the x and y axes. Conversely, if a line segment of fixed length R with its endpoints on the x- and y-axes is allowed to assume all possible positions, then the envelope formed by all of the line segments is an astroid (see Fig. 19.21(a)).

The astroid is also the envelope of the family of ellipses of the form $\frac{x^2}{a^2} + \frac{y^2}{(R-a)^2} = 1$, where the sum of their semi-major and semi-minor axes is R (see Fig. 19.21(b)).



(a) Using Tangent Lines (b) Using Ellipses

Figure 19.21. Drawing Astroids

19.7 Linking Hypocycloids and Epicycloids

The parametric equations for the hypocycloid and the epicycloid only differ in the sign associated with r. When the two radii are combined using $R + r$ then a epicycloid results. When its $R - r$, then an hypocycloid is created. The connection becomes clear is we think of the meaning of $-r$ in terms of the position of the rolling circle. A positive radius places it on the outside of the fixed circle while a negative moves it inside.

ehcycloid.py relaxes the constraints imposed by `epicycloid.py` and `hypocycloid.py` that R/r be positive, which makes it possible for one program to produce both types of cycloid. To keep the code short, only the cycloid curve is drawn without the circles or the animation (see Fig. 19.22).

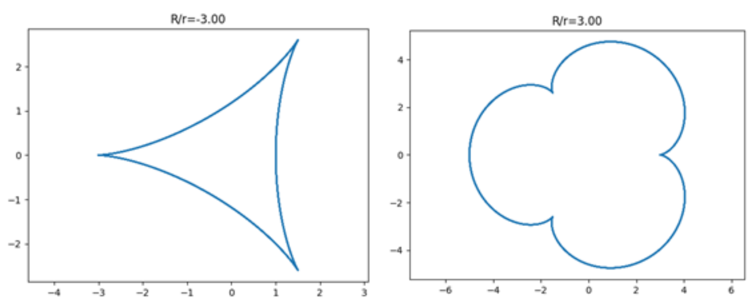


Figure 19.22. Hypocycloids and Epicycloids

19.8 Squaring the Circle

The problem of squaring the circle, namely constructing a square with the same area as a given circle using a straight edge and compass alone, is one of the classic problems of Greek mathematics. In 1882, Ferdinand von Lindemann proved that it wasn't possible if only those tools were available. However, it's quite straightforward if other mechanisms are permitted, such as a circle rolling along a line, to produce a cycloid as in Fig. 19.23.

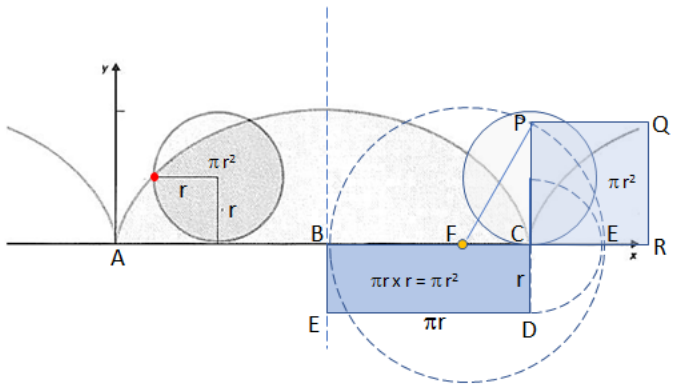


Figure 19.23. Squaring the Circle with a Cycloid

The (red) tracer point on the circle's circumference, moving from A to C describes a cycloid, meaning that the length of the straight line from A to C is equal to the circumference of the circle – that is, $2\pi r$.

Thus, if B is the midpoint of AC, then $BC = \pi r$. Hence, if $CD = r$, the area of the rectangle $BCDE$ is $\pi r \times r = \pi r^2$, which is also the area of the rolling circle. 'Squaring' this rectangle yields the square with the side PC. Thus the circle is also squared.

How is rectangle $BCDE$ created? Extend BC rightwards. Construct a circle with center C and radius r through D; it intersects BC at E. Find the midpoint F of BE. Construct a circle with diameter BE and center F. Extend the line CD until it meets that big circle at P. Construct a square on the segment CP, which we'll call $CPQR$.

How do we know that Area($BCDE$) = Area($CPQR$)? Consider:

$$\begin{aligned}
\text{Area}(CPQR) &= CP \cdot CR \\
&= CP^2 = FP^2 - FC^2 \\
&= (FP - FC)(FP + FC) \\
&= (FE - FC)(BF + FC) \quad \text{because } FP = FE = BF \\
&= CE \cdot BC = CD \cdot BC \\
&= \text{Area}(BCDE).
\end{aligned}$$

This means that the rolling circle, rectangle, and square in Fig. 19.23 all have the same area, πr^2, and we have successfully squared the circle.

Exercises

(1) Determine a $x = f(y)$ equation for the cycloid by eliminating θ from the parametric equations in Equs. 19.1.

(2) Show that the second derivative of the cycloid is $y'' = dy'/dx = -a/y^2$. Observe that this implies that the cycloid is concave down between the cusps.

(3) Show that the tangent to the cycloid at the point P in Fig. 19.24 passes through the top of the rolling circle.

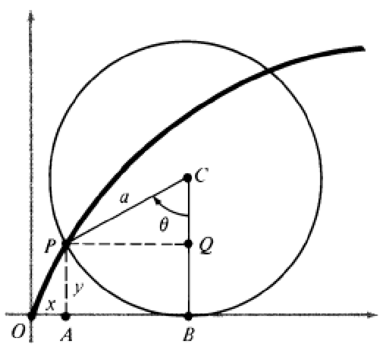


Figure 19.24. Cycloid Normal

(4) Assume that the circle in Fig. 19.24 rolls to the right along the x-axis at a constant speed, with the center C moving at v_0 units per second.

(a) Find the rates of change of the coordinates x and y of the point P.

(b) What is the greatest rate of increase of x, and where is P when this occurs?

(c) What is the greatest rate of increase of y, and for what value of θ is this attained?

(5) Find the area inside an astroid.

(6) Find the total length of an astroid.

(7) A hypocycloid with three cusps, $R = 3r$, is called a deltoid (see the lower left of Fig. 19.19). Find its parametric equations, and the total length of the curve.

Answers

(1) $P = (x, y)$ is $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$. Expressing θ and $\sin\theta$ in terms of y:

$$\cos\theta = 1 - \frac{y}{a}$$

$$\sin\theta = \sqrt{1 - \left(1 - \frac{y}{a}\right)^2}$$

$$= \sqrt{\frac{2y}{a} - \frac{y^2}{a^2}} = \frac{\sqrt{2ay - y^2}}{a}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{2ay - y^2}}{a}\right)$$

Substituting for θ and $\sin\theta$ in the expression for x:

$$x = a\left(\sin^{-1}\left(\frac{\sqrt{2ay - y^2}}{a}\right) - \frac{1}{a}\sqrt{2ay - y^2}\right)$$

$$a\sin^{-1}\left(\frac{\sqrt{2ay - y^2}}{a}\right) = x + \sqrt{2ay - y^2}$$

(2) The slope of the tangent to the cycloid: $y' = \cot\frac{\theta}{2}$.

$$\Longrightarrow \frac{dy'}{dx} = \frac{d}{d\theta}(\cot\frac{\theta}{2})/\frac{dx}{d\theta} = -\frac{1}{2}\csc^2\frac{\theta}{2}/\frac{dx}{d\theta}$$

$$= \frac{-\frac{1}{2}\csc^2\frac{\theta}{2}}{a(1 - \cos\theta)} = \frac{-1/(2\sin^2\frac{\theta}{2})}{a(1 - \cos\theta)}$$

$$= \frac{-1/(1 - \cos\theta)}{a(1 - \cos\theta)} = -\frac{1}{a(1 - \cos\theta)^2}$$

$$= -\frac{a}{y^2} \quad \text{because } y = a(1 - \cos\theta)$$

As $y \geq 0$ throughout, then $y' < 0$ wherever $y \neq 0$, which is at the cusps.

(3) The point at the top of the circle has coordinates $(a\theta, 2a)$. The slope of the tangent at P is given by $y' = \cot \frac{1}{2}t$. The equation of the tangent at P is therefore

$$y - a(1 - \cos\theta) = \frac{\sin\theta}{1 - \cos\theta}(x - a\theta + a\sin\theta).$$

We substitute $x = a\theta$ in this equation and solve for y, which gives

$$y = a(1 - \cos\theta) + \frac{\sin\theta}{1 - \cos\theta} \cdot a\sin\theta = \frac{a(1 - \cos\theta)^2 + a\sin^2\theta}{1 - \cos\theta} = 2a.$$

This shows that the tangent at P does indeed pass through the point $(a\theta, 2a)$ at the top of the circle.

(4) (a) The rate of change of x and y can be expressed as:

$$\frac{dx}{dt} = v_0(1 - \cos\theta) \qquad \frac{dy}{dt} = v_0 \sin\theta$$

where θ is the angle turned by C after time t.

Let the center of C be O. Without loss of generality, let P be at the origin at time $t = t_0$. By definition, P traces out a cycloid. $P = (x, y)$ is

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$$

Let (x_c, y_c) be the coordinates of O at time t. We have that $y_c = a$, which means that $x_c = v_0 t$. x_c is equal to the length of the arc of C that has rolled along the x-axis, so $x_c = a\theta$. So $\theta = \frac{v_0 t}{a}$, which implies $\frac{d\theta}{dt} = \frac{v_0}{a}$. Thus:

$$x = a\left(\frac{v_0 t}{a} - \sin\theta\right)$$

substituting for θ:

$$x = v_0 t - a\sin\theta$$

$$\begin{aligned} \implies \frac{dx}{dt} &= v_0 - a\cos\theta\frac{d\theta}{dt} = v_0 - a\cos\theta\frac{v_0}{a} \\ &= v_0(1 - \cos\theta) \end{aligned}$$

and:

$$y = a(1 - \cos\theta)$$

$$\begin{aligned} \implies \frac{dy}{dt} &= a\sin\theta\frac{d\theta}{dt} = a\sin\theta\frac{v_0}{a} \\ &= v_0\sin\theta \end{aligned}$$

(b) The maximum rate of change of x is $2v_0$, which happens when P is at the top of the circle C. The rate of change of x is given by:

$$\frac{dx}{dt} = v_0(1 - \cos\theta)$$

This is a maximum when $1 - \cos\theta$ is a maximum, so when $\cos\theta$ is at a minimum. That happens when $\cos\theta = -1$, so $\theta = \pi, 3\pi, \ldots$. That is, when $\theta = (2n+1)\pi$ where $n \in \mathbb{Z}$. That is, when P is at the top of the circle C. When $\cos\theta = -1$ we have:

$$\frac{dx}{dt} = v_0(1 - (-1)) = 2v_0$$

(c) The maximum rate of change of y is v_0, which happens when $\theta = \frac{\pi}{2} + 2n\pi$ where $n \in \mathbb{Z}$. The rate of change of y is given by $\frac{dy}{dt} = v_0 \sin\theta$.
This is a maximum when $\sin\theta$ is a maximum, when $\sin\theta = 1$. That happens when $\theta = \frac{\pi}{2} + 2n\pi$ where $n \in \mathbb{Z}$. When $\sin\theta = 1$ we have $\frac{dy}{dt} = v_0$.

(5) We'll show that the area inside an astroid constructed within a circle of radius a (see Fig. 19.25) is $A = \frac{3\pi a^2}{8}$. Locate the astroid H with its center at the origin and its cusps positioned on the axes.

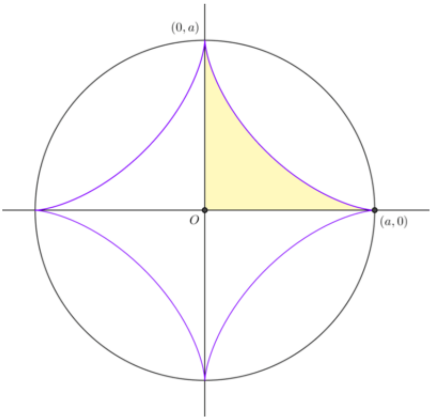


Figure 19.25. Astroid Area

By symmetry, it's sufficient to evaluate the area shaded yellow and to multiply it by 4. The astroid equation is:

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta.$$

Thus:

$$A = 4\int_0^a y\,dx \quad = 4\int_{x=0}^{x=a} y\frac{dx}{d\theta}\,d\theta$$

Differentiate x with respect to θ:

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

Substitute into A to give:

$$A = 4\int_{x=0}^{x=a} a\sin^3\theta\cdot 3a\cos^2\theta(-\sin\theta)\,d\theta$$

When $x = a$, $a\cos^3\theta = a$; when $x = 0$, $\theta = \frac{\pi}{2}$; when $x = a$, $\theta = 0$. Simplifying

$$\begin{aligned} A &= 4\int_{\theta=0}^{\theta=\pi/2} a\sin^3\theta\cdot 3a\cos^2\theta(-\sin\theta)\,d\theta \\ &= 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta \end{aligned}$$

Simplifying the integrand:

$$\begin{aligned} \sin^4\theta\cos^2\theta = \frac{(2\sin\theta\cos\theta)^2}{4}\sin^2\theta &= \frac{\sin^2 2\theta}{4}\sin^2\theta \\ &= \frac{\sin^2 2\theta}{4}\cdot\frac{1-\cos 2\theta}{2} \\ &= \frac{\sin^2 2\theta - \sin^2 2\theta\cos 2\theta}{8} \\ &= \frac{1-\cos 4\theta}{16} - \frac{\sin^2 2\theta\cos 2\theta}{8} \end{aligned}$$

Thus:

$$
\begin{aligned}
A &= 12a^2 \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{16} - \frac{\sin^2 2\theta \cos 2\theta}{8} \right) d\theta \\
&= \frac{3a^2}{4} \int_0^{\pi/2} (1-\cos 4\theta)\, d\theta - \frac{3a^2}{2} \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta \\
&= \frac{3a^2}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} - \frac{3a^2}{2} \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta \\
&= \frac{3a^2}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} - \frac{3a^2}{2} \left[\frac{\sin^3 2\theta}{6} \right]_0^{\pi/2} \\
&= \frac{3a^2}{4} \left(\frac{\pi}{2} - \frac{\sin 2\pi}{4} \right) - \frac{3a^2}{2} \cdot \frac{\sin^3 \pi}{6} \\
&= \frac{3\pi a^2}{8} - \frac{3a^2}{16} \sin 2\pi - \frac{3a^2}{12} \sin^3 \pi = \mathbf{\frac{3\pi a^2}{8}}
\end{aligned}
$$

(6) We'll show that the total length of the four arcs of an astroid constructed within a deferent of radius a (see Fig. 19.26) is $L = 6a$. Once again the astroid H has its center at the origin and its cusps positioned on the axes.

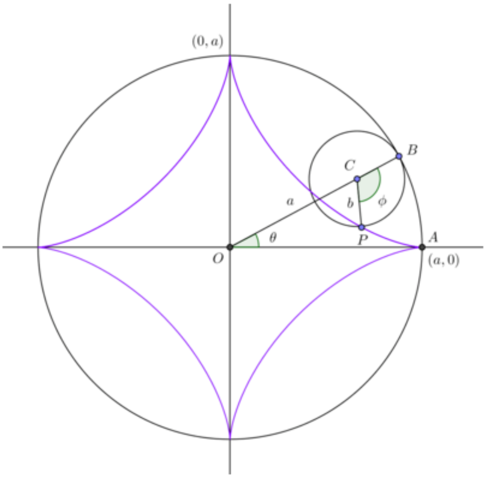


Figure 19.26. Length of an Arc of an Astroid

L is 4 times the length of one arc of the astroid, and the arc length is defined as:

$$L = 4\int_{\theta=0}^{\theta=\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

where, from the equation for the astroid:

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

we have:

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Thus:

$$\begin{aligned}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} &= \sqrt{9a^2(\sin^4\theta\cos^2\theta + \cos^4\theta\sin^2\theta)}\\ &= 3a\sqrt{\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)}\\ &= 3a\sqrt{\sin^2\theta\cos^2\theta} = 3a\sin\theta\cos\theta\\ &= \frac{3a\sin 2\theta}{2}\end{aligned}$$

Thus:

$$\begin{aligned}L &= 4\int_0^{\pi/2} \frac{3a}{2}\sin 2\theta\, d\theta\\ &= 6a\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = 6a\left(-\frac{\cos\pi}{2} + \frac{\cos 0}{2}\right)\\ &= 6a\left(-\frac{(-1)}{2} + \frac{1}{2}\right) = \mathbf{6a}\end{aligned}$$

(7) By definition, a deltoid is a hypocycloid with 3 cusps.

Let H be the deltoid generated by the epicycle C_1 of radius b rolling without slipping around the inside of a deferent C_2 of radius $a = 3b$ (see Fig. 19.27). Let C_2 have its center located at the origin. Let P be a point on the circumference of C_1. Let C_1 be initially positioned so that P is its point of tangency to C_2, located at point $A = (a, 0)$ on the x-axis. Let (x, y) be the coordinates of P as it travels over the plane. The point $P = (x, y)$ is described by the parametric equation:

$$x = 2b\cos\theta + b\cos 2\theta, \qquad y = 2b\sin\theta - b\sin 2\theta$$

where θ is the angle between the x-axis and the line joining the origin to the center of C_1.

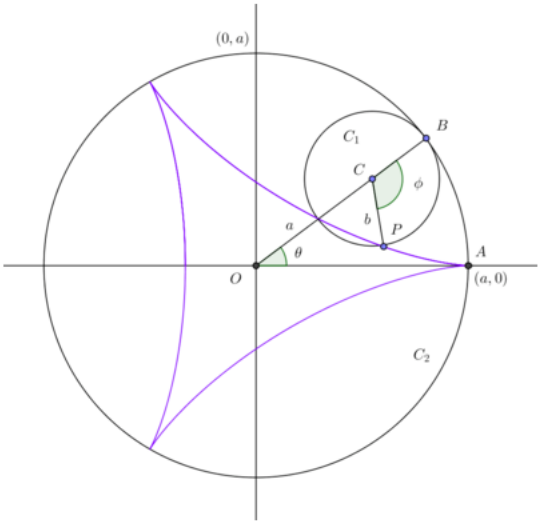


Figure 19.27. A Deltoid

Using the hypocycloid equation, H is given by:

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$
$$y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

This can be generated by an epicycle C_1 of radius $\frac{1}{3}$ the radius of the deferent. Thus $a = 3b$ and the equation of H is now given by:

$$x = 2b\cos\theta + b\cos 2\theta, \qquad y = 2b\sin\theta - b\sin 2\theta$$

We'll now show that the total length of the arcs of a deltoid constructed within a deferent of radius a is $L = \frac{16a}{3}$.

Let one of H's cusps be positioned at $(a, 0)$, and note that L is 3 times the length of one arc of the deltoid. The arc length equation is:

$$L = 3\int_{\theta=0}^{\theta=2\pi/3} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

where, the deltoid is:

$$x = 2b\cos\theta + b\cos 2\theta, \qquad y = 2b\sin\theta - b\sin 2\theta$$

and so:

$$\frac{dx}{d\theta} = -2b\sin\theta - 2b\sin 2\theta, \qquad \frac{dy}{d\theta} = 2b\cos\theta - 2b\cos 2\theta$$

Thus:

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (-2b\sin\theta - 2b\sin 2\theta)^2 + (2b\cos\theta - 2b\cos 2\theta)^2 \\
&= 4b^2\left((-\sin\theta - \sin 2\theta)^2 + (\cos\theta - \cos 2\theta)^2\right) \\
&= 4b^2(\sin^2\theta + 2\sin\theta\sin 2\theta + \sin^2 2\theta + \cos^2\theta - 2\cos\theta\cos 2\theta + \cos^2 2\theta)
\end{aligned}$$

Using the sum of squares for sine and cosine:

$$\begin{aligned}
&= 4b^2(2 + 2\sin\theta\sin 2\theta - 2\cos\theta\cos 2\theta) \\
&= 8b^2(1 + \sin\theta\sin 2\theta - \cos\theta\cos 2\theta)
\end{aligned}$$

and the double angle formula for sine

$$= 8b^2(1 + 2\sin^2\theta\cos\theta - \cos\theta\cos 2\theta)$$

and then the double angle formula for cosine:

$$= 8b^2(1 + 2\sin^2\theta\cos\theta - \cos\theta(1 - 2\sin^2\theta))$$

Simplifying

$$\begin{aligned}
&= 8b^2(1 - \cos\theta + 4\sin^2\theta\cos\theta) \\
&= 8b^2(1 - \cos\theta + 4\cos\theta(1 - \cos^2\theta))
\end{aligned}$$

Use the sum of squares for sine and cosine again, and the difference of two squares:

$$\begin{aligned}
&= 8b^2(1 - \cos\theta + 4\cos\theta(1 + \cos\theta)(1 - \cos\theta)) \\
&= 8b^2(1 - \cos\theta)(1 + 4\cos\theta(1 + \cos\theta)) \\
&= 8b^2(1 - \cos\theta)(1 + 4\cos\theta + 4\cos^2\theta) \\
&= 8b^2(1 - \cos\theta)(1 + 2\cos\theta)^2
\end{aligned}$$

Use the half angle formula for sine:

$$\begin{aligned}
&= 8b^2\left(2\sin^2\frac{\theta}{2}\right)(1 + 2\cos\theta)^2 \\
&= 16b^2\sin^2\frac{\theta}{2}(1 + 2\cos\theta)^2
\end{aligned}$$

Thus:

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 4b\sin\frac{\theta}{2}|1 + 2\cos\theta|$$

In the range 0 to $\frac{2\pi}{3}$, $1 + 2\cos\theta$ is not less than 0, and so:

$$L = 3\int_0^{2\pi/3} 4b \sin\frac{\theta}{2}(1 + 2\cos\theta)\,d\theta$$

Put $u = \cos\frac{\theta}{2}$ so $2\frac{du}{d\theta} = -\sin\frac{\theta}{2}$. As θ increases from 0 to $\frac{2\pi}{3}$, u decreases from 1 to $\frac{1}{2}$. Then with the half angle formula for cosine

$$1 + 2\cos\theta = 1 + 2\left(2\cos^2\frac{\theta}{2} - 1\right) = 4u^2 - 1$$

Substituting $2\frac{du}{d\theta} = -\sin\frac{\theta}{2}$ and change the limits of integration from $\theta = 0 - \frac{2\pi}{3}$, to $u = 1 - u = \frac{1}{2}$, and after dealing with the sign:

$$\begin{aligned}
L &= 12b\int_1^{1/2} (1 - 4u^2)(2)\,du \\
&= 24b\left[u - \frac{4}{3}u^3\right]_1^{1/2} \\
&= 24b\left(\left(\frac{1}{2} - \frac{4}{3}\cdot\frac{1}{8}\right) - \left(1 - \frac{4}{3}\right)\right) \\
&= 24b\left(\frac{2}{3}\right) = 16b = \mathbf{\frac{16a}{3}}
\end{aligned}$$